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Zoom in: Exploring perceptions of the multiplication symbol (×) up close

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Abstract.
This study explored the perceptions of 51 prospective elementary school teachers selected using purposive sampling technique regarding the multiplication symbol (×) in arithmetic operations. Using a qualitative hermeneutic phenomenological approach, data were collected through task-based interviews administered to education students specializing in elementary education. The tasks were designed to explore candidates' understanding and interpretation of the '×' symbol, including uncovering conceptual images, understanding the '×' symbol in different contexts, ability to represent the '×' symbol, flexibility in relating the '×' symbol among concepts, and problem-solving skills. Data analysis involved thematic coding and interpretive analysis to uncover patterns and insights into candidates' cognitive frameworks. The findings revealed significant variation in candidates' understanding of the '×' symbol, influenced by their educational background and personal experiences with mathematics. This study highlights the need for improved mathematics instruction and curriculum design to equip future teachers with a deep and accurate understanding of arithmetic symbols, which is critical for effective mathematics teaching at the elementary level.

Keywords:
Arithmetic operation; concept image; multiplication symbol; phenomenological hermeneutics; primary school teacher candidates

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INTRODUCTION

Conception in mathematics education is a subjective internal understanding that a person has about a concept, referring to the entire group of internal representations and associations that are triggered by a concept. This understanding involves various cognitive and emotional elements that shape the way a person understands and uses the concept in the context of mathematics and everyday life. Sfard (1991) describes a conception as a collection of representations and internal associations that shape a person's understanding of a concept. This understanding is subjective and unique to each individual because it is influenced by their personal experiences and educational background. Tall and Vinner (1981) introduced the term concept image which includes everything that is in a person's mind related to a concept. This description can vary greatly between individuals and does not always align with formal or conceptual definitions of the concept. They also introduced the term concept definition to refer to a formal and standardized way of describing a concept, often used in academic and educational contexts to ensure uniformity in understanding.

Chin and Pierce (2019) stated that students' understanding of concepts is often based on subjective personal conceptions and influenced by their experiences. This can cause difficulties in achieving a standardized form of understanding, even though there are mathematical rules agreed upon by the mathematics community. These personal conceptions can be supportive or hindering in understanding new situations. For example, the multiplication symbol (×) is often interpreted

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as "repeated addition" in the context of integers. In this case, students may understand multiplication as the addition of the same number, such as considering 3×4 as $4 + 4 + 4$. This conception is quite supportive when working with integers and provides a good foundation for understanding basic mathematical operations. However, when students are faced with the context of multiplication in the form of fractions, this conception of repeated addition becomes inadequate. As an illustration, in the case of the multiplication of $\frac{1}{2} \times \frac{1}{3}$, it makes no sense to interpret this operation as repeated addition of fractional numbers, because it cannot be expressed as simple repeated addition.

In addition, the conception formed in students is greatly influenced by the concept image held by the teacher. If the teacher's concept image is inaccurate or not deep enough, the understanding transferred to students also tends to be inaccurate or limited. Therefore, ensuring that the teacher's concept image is appropriate is very important, because the teacher is the main mediator in the learning process and student understanding. As expressed by Sfard (1991), the teacher's understanding of a concept directly shapes and influences how students learn and understand the concept. Shulman (1986) also emphasized the importance of teachers' pedagogical content knowledge, which includes a deep understanding of the concepts they teach and the ability to communicate them effectively to students. Therefore, developing and improving the teacher's concept image not only improves the quality of teaching but also ensures that students receive correct and deep representations and understanding of concepts. Without the right concept image, students can experience misconceptions that can affect their future mathematics performance (Ardiansari et al., 2023)

The symbol '×' in arithmetic operations plays an important role in learning in elementary schools because it is one of the fundamental symbols in mathematics (Van de Walle et al., 2019). The symbol '×' is a very flexible symbol in arithmetic operations where its use can include various meanings according to the context such as repeated addition, multiplication, transformation, as well as more complex applications such as in algebra, fractions and geometry. A good understanding of the meaning and use of these symbols is essential for understanding basic and advanced concepts in mathematics.

Investing in ensuring teacher candidates understand the symbol '×' in operation is a critical step in ensuring students can build a strong mathematical foundation (Hill et al., 2005). If prospective teachers do not have a correct understanding of these symbols, they may convey incorrect or unclear information to students which can lead to ongoing misunderstandings and difficulties in further learning mathematics (Ma, 1999). Additionally, understanding how this symbol is used in various contexts, such as understanding that $\frac{1}{2} \times \frac{1}{3}$ is not a repeated addition but represents a part of a whole, is critical to ensuring that teacher candidates can help students transfer understanding from one context to another (Thompson, 1994).

Several previous studies have discussed the symbol "×", such as Sfard (1991) explored how mathematical symbols, including the multiplication symbol, are conceptualized as processes and objects. This provides insight into how students perceive and interpret symbols in different contexts; Gray and Tall (1994) introduced the concept of "procept" which is a combination of process and concept, to explain how learners may struggle to understand symbols such as "×" due to its dual nature. Their findings highlight the challenges that students face in understanding mathematical symbols; Núñez et al. (1999) discussed how the physical and cognitive aspects of learning affect the understanding of mathematical symbols, including "×". This suggests that the internalization of mathematical concepts can affect how symbols are interpreted; MacGregor & Stacey (1997) examined how students of different age groups interpret algebraic symbols, including the multiplication symbol. This identified common misconceptions and suggested teaching strategies to overcome these challenges; Kieran (1989) focused on the structural aspects of algebra learning and how students interpret symbols such as "×" in algebraic expressions. This provides an in-depth analysis of the cognitive processes involved in understanding these symbols; Steinle and Stacey (2004) investigated how misconceptions about mathematical symbols persist over time, particularly in the context of decimals and multiplication. This highlights the importance of addressing these misconceptions early in the learning process; also, a study conducted by Önal (2023) found that elementary school students often recognize the multiplication symbol "×", but have difficulty applying it to unfamiliar word problems or contexts. Their understanding tends to be limited to the notion of multiplication as repeated addition, without a deeper understanding of the broader concept.

While Sfard (1991), Gray and Tall (1994), and Núñez et al. (1999) focused on how students conceptualize the multiplication symbol "×" as a process and object, and the associated cognitive challenges, this study highlights the unique perspective of preservice elementary teachers in understanding this symbol. Unlike previous studies that have generally focused on student understanding, my research explores how preservice teachers interpret and teach the symbol "×," taking into account the pedagogical challenges that arise from their own understandings. This provides new insights into how preservice teachers' misconceptions may influence their future teaching, a perspective that has not been widely explored in previous literature.

This research highlights the unique perspective of primary school teacher candidates on the symbol '×' in arithmetic operations, an area that has not been widely explored in the academic literature. This research uses a hermeneutic phenomenological approach to reveal how primary school teacher candidates interpret the symbol '×' and how this understanding has the potential to influence their teaching methods. The Hermeneutic Phenomenological Approach combines phenomenology and hermeneutics to understand experience and meaning from the perspective of the subject. Phenomenology focuses on subjective experience and the essence of the phenomenon without external distortion, while hermeneutics focuses on interpreting the meaning of a text or experience, taking into account context and background. This approach offers a new contribution by exploring the conceptual and representational dimensions of the symbol '×' that may not have been detected in previous research that focused more on student learning outcomes or conventional teaching strategies, while less exploring prospective teachers' understanding of this symbol from a conceptual and representational perspective. Moreover, little research links understanding the symbol '×' with prospective teachers' readiness to teach and application in real teaching contexts in elementary schools. Therefore, this research seeks to fill this gap by exploring prospective teachers' understanding of the symbol '×', and how this understanding can influence their teaching strategies in the future.

METHOD

Research Design

The research was carried out using a hermeneutic phenomenological approach to explore prospective elementary school teacher students' understanding of the symbol '×' in arithmetic operations, as well as attempting to reveal their conceptual image of the fundamental meaning attached to this symbol. In achieving this goal, the following two questions were used: (a) What is the concept image of primary school teacher candidates regarding the symbol '×' in arithmetic operations in the context of mathematics learning? (b) How do their personal experiences influence this understanding?

In this research, to ensure the validity of the qualitative data collected through task-based interviews given to specialist basic education students, a series of credibility and validity tests were carried out. The credibility of the data was tested using triangulation techniques, namely by comparing interview results with other data sources such as observation and literature studies, as well as through member checking where respondents were given the opportunity to confirm and clarify the results of the interpretation made by the researcher. Additionally, transferability is enhanced by providing detailed context descriptions, allowing readers to evaluate the applicability of the findings in different contexts. Research dependability is achieved through an audit trail, which involves complete documentation of the research process so that analytical procedures can be replicated or reviewed by other researchers. To ensure confirmation, peer debriefing and consensus between researchers were carried out to ensure that the results of data interpretation reflected an objective and unbiased understanding. Thematic coding techniques and interpretative analysis were used to identify consistent patterns and insights regarding students' cognitive frameworks and understanding of arithmetic symbols and operations, ensuring that the research results were valid and accountable.

Setting and Participants

This research was conducted at a private university located in Probolinggo Regency, East Java, Indonesia. This university is known to have an Elementary School Teacher Education study program that is the focus of this research. The university environment provides a supportive academic atmosphere, with adequate facilities for teaching and learning activities and research. The university where the research was conducted has a good reputation for producing competent educators, especially in the field of elementary education. This study involved 51 students who were studying in the Elementary School Teacher Education program. These participants came from various backgrounds, both in terms of age, gender, and previous educational experience. They were prospective 6th semester teachers who were preparing themselves for teaching practice at the elementary school level in the following semester.

A total of 51 6th semester PGSD students were selected as participants in this study because they were in the final stage of their study program and were expected to have a mature understanding of basic mathematical concepts, including the multiplication symbol "×". In addition, they had gained sufficient learning experience to be able to provide relevant perspectives on how the symbol should be taught to elementary school students. These students voluntarily stated their willingness to participate in the study. Their participation was invaluable in revealing their perceptions and understanding of the '×' symbol in arithmetic operations. Throughout the research, the anonymity of the participants was strictly maintained. Their identities were disguised to maintain the confidentiality of personal information and ensure that the data collected was free from bias or external pressure.

All participants underwent the entire research process through task-based interviews designed to explore their understanding and interpretation of various arithmetic operation symbols. The interviews were conducted in a comfortable and supportive atmosphere, either in a classroom or in another appropriate area within the university. Before the interview began, participants were given a detailed explanation of the purpose of the study, the data collection process, and their right to withdraw from the study at any time without negative consequences. The diversity of participants in terms of educational background, teaching experience, and perception of mathematics provides a broad and deep perspective on how arithmetic operation symbols are understood and translated in the learning context in elementary schools. It is hoped that this diversity can enrich research results and provide a comprehensive picture of the understanding of arithmetic symbols among prospective teachers in the area.

Instrument

This research instrument uses task-based interviews as a method for collecting data. This interview is designed to test students' understanding of mathematical concepts, especially in the context of arithmetic operation symbols. Each participant will be given 5 questions that include components about (a) conceptual meaning, (b) understanding symbols in various contexts, (c) symbol representation, (d) connections between concepts, and (e) problem solving skills, as indicated by Table 1.

The choice of a task-based interview instrument to explore the understanding of the '×' symbol in prospective elementary school teachers was based on several basic reasons. First, this method allows researchers to see how participants apply their theoretical knowledge in practical contexts which is important for understanding how they process and interpret the symbol '×' in various situations (Mejía-Ramos & Weber, 2020). Through the concrete tasks given, participants can demonstrate their understanding of the symbol '×' not only as a multiplication operation but also in broader contexts such as repetition, scale, and distribution (Hurst, 2007). Second, task-based interviews allow researchers to dig deeper into the thought processes and problem-solving strategies used by participants. In this way, researchers can identify whether participants have deep conceptual understanding or only limited procedural understanding. Third, this instrument provides an opportunity for direct interaction that allows clarification and further exploration of participants' responses, thereby ensuring the data obtained is richer and more comprehensive (Van Es & Sherin, 2008). Therefore, task-based interviews are the right instrument to gain in-depth and detailed insight into prospective teachers' understanding of the symbol '×'.

Table 1. Task-based interview

Problem Type	Problem Description	Interview Question
Conceptual Meaning	Define the meaning of the symbol '×' in arithmetic operations.	Does the symbol '×' only refer to one meaning? If yes, what is it? If not, please state. Why can this symbol only be interpreted that way? Why can one symbol have several meanings? Give an example.
Symbol Understanding in Various Contexts	What does the symbol '×' mean in each of the following operations? a) 3×4, -2×3, $\frac{1}{2} \times \frac{1}{3}$	"In arithmetic operations, do you see any conceptual difference between '3×4' and '4×3'? If so, how do you explain it to students? Otherwise, why do you think they are the same?"
Symbol representation	How do you represent 3×6 in four ways different (Array, Equal Group, Repeated Addition, and Number Line)?	How would you explain the 3×6 representation using each of these methods to elementary school students? How does this help them understand the concept of multiplication? Are there other ways that can be used to represent it besides these four ways?
Connections Between Concepts	Which one is correct? Explain why. $a^2 \times a^3 = a^5$ or $a^2 \times a^3 = a^6$	How do you interpret the use of the symbol '×' in this context? How do you apply the concept of multiplying exponentials to determine the correct answer to this problem?
Problem Solving Skills	Calculate the following mixed operations: $12 \times 13 + 13 \times 14 - 14 \times 15 + 15 \times 16$	What steps did you take to find the correct values? How can you ensure that the values you find are correct? Describe the verification process you use.

RESULTS AND DISCUSSION

In this section, the discussion focuses on analyzing the concept image of prospective elementary school teacher students regarding the symbol '×' in arithmetic operations. Concept image is the overall mental structure and understanding a person has of a concept that is formed through learning experiences, social interactions, and applications in everyday life (Tall & Vinner, 1981). This analysis aims to explore the in-depth understanding and variations in interpretations that students have regarding the symbol '×' and how this symbol is perceived and used in various mathematics learning contexts. The presentation will be carried out based on each problem.

The following are the details of the main findings of this study, which describe the understanding and errors experienced by students related to mathematical concepts and operations, especially in the use of the symbol '×'. These findings were taken from respondents consisting of 51 prospective teacher students. a) The symbol '×' is understood as a multiplication sign that is limited to repetition or the total number of the same group; b) Does not see the conceptual difference between 3 × 4 and 4 × 3 because both produce 12; c) Difficulty explaining multiplication with negative numbers, only remembering the negative result without understanding the concept; d) Able to perform multiplication operations on fractions, but does not understand the meaning of the symbol '×' in the context of fractions; e) Difficulty correctly depicting multiplication arrays; f) Difficulty representing multiplication on the number line correctly; g) Errors in exponent rules; h) Errors in the order of operations in following the correct mathematical procedure.

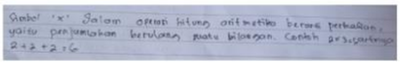
Problem 1

This problem is used to explore the conceptual understanding of prospective elementary school teacher students regarding the symbol '×' by considering several reasons. First, this question forces students to articulate their understanding of the basic concept of multiplication which is often considered simple, but actually has many layers of meaning that can vary depending on the context of its use in basic arithmetic, algebra, and even in geometric representations. According to Rittle-Johnson and Alibali (1999), conceptual understanding is the key to building a deeper knowledge base that can help students apply concepts correctly in various mathematical situations and problems. In addition, Vinner (1983) showed that a strong understanding of mathematical symbols allows individuals to use those concepts flexibly and adaptively in a variety of problem-solving situations. By asking students to define the meaning of the symbol '×', this can be used to measure the extent to which they understand that this symbol not only shows repeated addition, but also functions as an operator that has various meanings in different contexts, such as distribution, proportion, and even the relationships between variables in algebraic equations. This is in accordance with the views of Hiebert and Carpenter (1992) who emphasize that deep conceptual understanding is essential for building coherent and applicable mathematical knowledge.

The results of the research show that of the 51 students who were research subjects, all of them stated that the symbol '×' is a symbol for multiplying two numbers which is limited to the context as a times symbol which shows the multiplication operation with an emphasis on repetition or the total number of the same group (Amarasinghe et al., 2013; Burns, 2001; Haylock & Manning, 2014). For example, 2×5 as “two groups of five” which means five added twice i.e. $5 + 5 = 10$. In this context, they see '×' only as a tool to calculate the result of two numbers being multiplied, without considering the concept involved. wider or other applications of multiplication. Figure 1 shows examples of student answers along with their translation into English.

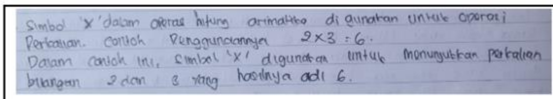
This understanding approach can be categorized as operational view or operational concept image which shows that their understanding of the '×' symbol is limited to basic operations without looking at deeper concepts or the application of multiplication in a wider context. Findings from previous studies such as (Chin & Pierce, 2019; Lee et al., 2021) found that many prospective teachers only understand the symbol '×' as a repetition of addition. This is consistent with the finding that prospective elementary school teacher students in this study also had a limited operational view.

Although this understanding is important for introducing the basic concept of multiplication to students, viewing it solely as a times symbol is not enough to prepare student teachers in teaching broader mathematical concepts to elementary school students because the symbol '×' also has broader meanings such as multiplication symbols in more complex contexts, including multiplication of fractional numbers, where the concept of repetition does not apply directly. For example, the multiplication $\frac{1}{2} \times \frac{1}{3}$ is not repeated addition, but the product of two fractions that gives a new value that cannot be explained simply by repeated addition. Furthermore, the symbol '×' is also used in geometric contexts to calculate areas, such as length times width to determine the area of a rectangle, which is different from the concept of repeated addition. Therefore, a narrow understanding of the symbol '×' as a times symbol is not enough to support comprehensive mathematics teaching to elementary school students. Student teachers need to understand the various meanings of these symbols in various contexts, so that they can teach mathematical concepts more effectively and in depth.



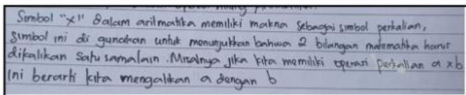
The symbol '×' in arithmetic operations means multiplication or addition of a number repeatedly. For example, 2×3=6 means 2+2+2=6

(a)



The symbol '×' in arithmetic operations is used for multiplication operations. Example of use $2 \times 3 = 6$ In this example, the symbol '×' is used to show the multiplication of the numbers 2 and 3 which results in 6.

(b)

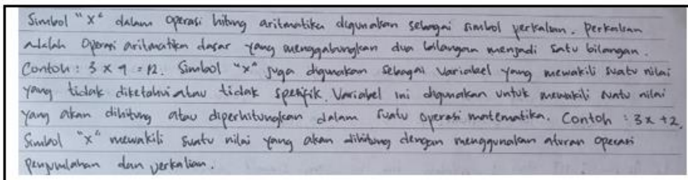


The symbol '×' in arithmetic operations is used for multiplication operations, this symbol is used to indicate that 2 mathematical numbers must be multiplied by each other. For example, we have the multiplication operation $a \times b$, this means we multiply a by b.

(c)

Figure 1. Student teacher candidates' responses to problem 1

When responding to the question "does the symbol '×' only refer to one meaning?" It is not surprising that students generally answer with 'yes', namely as a symbol of multiplication. However, 10 of the 51 student respondents answered that the symbol '×' not only refers to multiplication but also 'variables' as shown in Figure 2.



The symbol '×' in arithmetic operations is used as a multiplication symbol. Multiplication is a basic arithmetic operation that combines two numbers into one number, for example: 3×4=12. The symbol '×' is also used as a variable to represent an unknown or unspecified value. This variable is used to represent a value that will be calculated or taken into account in a mathematical operation, for example $3x + 2$, the symbol '×' represents a value that will be calculated using addition and multiplication operations.

Figure 2. Example of an answer with the symbol '×' meaning a variable

This is surprising because variables should be represented by the letter 'x' not the symbol '×'. This misunderstanding is known as "symbol overgeneralization" which is when someone generalizes or applies the meaning of a symbol excessively outside its original context (Taber & Akpan, 2017). In this context, students seem to blur the difference between the symbol '×' as a sign of the multiplication operation and the letter 'x' as a variable used in algebra. This misunderstanding can be caused by their exposure to various symbolic representations that were not clearly differentiated in previous teaching or through learning experiences that did not pay enough attention to the context of symbol use (Chick, 2007). Additionally, the lack of emphasis on the contextual distinction between operation symbols and variables in the curriculum may also contribute to this confusion (Moschkovich, 1999; Nunes et al., 2016).

Problem 2

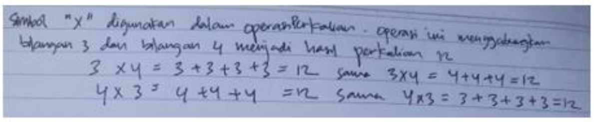
The symbol '×' has different meanings depending on the context in which it is used in mathematics. In basic arithmetic operations, this symbol is often interpreted as a simple multiplication operation between two integers. However, this meaning expands when applied to other contexts such as multiplying fractions, decimals, or even in algebraic and geometric operations. Understanding these various contexts allows prospective teachers to teach mathematics in a more holistic and meaningful way (Skemp, 1987; Wu, 2011). Problem 2 is designed to help student teachers identify and differentiate the applications of the symbol '×' in basic arithmetic, such as operations with whole numbers, operations with negative numbers, and operations with fractions. This deep understanding is important for teaching students in a comprehensive and contextual way.

Problem (a) 3×4 explores the basic concept of multiplication known as repeated addition. Multiplication is often understood as adding the same number repeatedly, as stated by Leung and Cheung (1988), namely multiplication can be understood as repeated addition, where the multiplicand is added a number of times specified by the multiplier as illustrated by the Figure 3. For example, 4 multiplied by 3, often written as 3×4 and spoken as "3 times 4", can be calculated by adding 3 copies of 4 together: $3 \times 4 = 4 + 4 + 4 = 12$. Here, 3 (the multiplier) and 4 (the multiplicand) are the factors, and 12 is the product.

$$a \times b = \underbrace{b + \cdots + b}_{a \text{ times}}.$$

Figure 3. Illustration of multiplication as repeated addition

All respondents said they did not see a conceptual difference between 3×4 and 4×3 because both are the product of two positive integers (3 and 4) which both produce 12 as shown in Figure 4.



The symbol '×' is used in the multiplication operation. This operation combines the number 3 and number 4 to form the product 12.

$3 \times 4 = 3 + 3 + 3 + 3 = 12$ equals $3 \times 4 = 4 + 4 + 4 = 12$

$4 \times 3 = 4 + 4 + 4 = 12$ equals $4 \times 3 = 3 + 3 + 3 + 3 = 12$

Figure 4. Example of a response showing the similarities between 3×4 and 4×3

Between 3×4 and 4×3 it looks the same when looking at the result or product of the multiplication, namely 12. This same result is guaranteed by the commutative property of the multiplication operation, which states that the order of the factors in the multiplication does not affect the final result, hence the concept of $3 \times 4 = 4 \times 3 = 12$ (NCTM, 2010; NIST, 2020). However, there are fundamental differences between the multiplicand and the multiplier of the two that are often overlooked. However, this difference can be very important. In practical situations such as administering drug doses, for example, when it is written 3 x 1 capsule a day, the meaning is certainly different if it is written 1 x 3 capsules a day. Even though the number of capsules that enter the body is the same, namely 3 capsules, the dose size certainly has a serious impact on the patient's health (Ryan et al., 2014). In addition, in the context of mathematics learning, a deep understanding of the role of each factor in multiplication can help students apply this concept in a variety of more complex situations. For example, in solving problems involving area calculations, knowing the difference between length and width and how they interact through multiplication operations is essential to obtain accurate results (Baroody, 2006; Caron, 2007). Misunderstandings in this context can lead to errors in estimation or measurement which in turn impact the final results in the project or experiment. It is important to master this basic difference because student teachers must be able to explain this basic concept to elementary school students who are new to multiplication.

In the second part, -2×3, this problem introduces the concept of multiplication involving negative numbers, showing that the symbol '$\times$' can change the sign of the product when one of the factors is negative. This shows that multiplication is not only about repetition but also about how the sign of the number affects the final result, which is a critical understanding for operations with negative numbers (Devlin, 2012). This understanding is important for prospective teachers to teach students about the influence of signs in arithmetic operations.

Multiplication can generally be understood as repeated addition. For example, 2×3 can be interpreted as 3 added twice, i.e., $3 + 3 = 6$. However, when negative numbers are involved, this interpretation needs to be expanded to consider how the negative sign affects the repetition. To understand -2×3, it can be thought of as a subtraction of the corresponding positive number. That is, -2×3 is the same as reversing the direction of two repetitions of 3, which results in the negative amount of what would be obtained from 2×3. Mathematically it can be written: $-2 \times 3 = -(2 \times 3) = -6$. So, the interpretation is that -2×3 gives the same result as 2×3, but with a negative sign. In another context, this could be interpreted as taking the value 3 and adding it in the negative direction twice (Kline, 1982).

Students have difficulty explaining the meaning of multiplication with negative numbers and tend to answer only based on memory that the result of multiplying negative and positive numbers is a negative number. They don't understand the concepts behind the rules, such as how the negative sign affects the repetition operation in the context of multiplication. This can be seen from the response when asked to explain the meaning of multiplication -2×3, all students in this study only said that "because -2 is negative, the result is -6".

In the third part, $\frac{1}{2} \times \frac{1}{3}$, the symbol '$\times$' is applied to the multiplication of different fractions by repetition or addition of groups of the same size, but rather represents 'taking one part from another'. This shows that the symbol '$\times$' in the context of a fraction is used to calculate parts of a value which introduces the concept of proportion and ratio (Lamon, 2001; McMullen & Hoof, 2020). It is very important to understand multiplication in the context of proportions and ratios which are often applied in various scientific disciplines and in everyday life.

To explain why $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, it is necessary to understand that multiplying fractions is an operation that involves simplifying the product of the numerator and denominator of each fraction. When multiplying $\frac{1}{2} \times \frac{1}{3}$, it is actually similar to finding how many small parts are formed when taking half from a third (Devlin, 2012). Mathematically, multiplying two fractions is done by multiplying the numerator (top) of one fraction by the numerator of the other fraction, and multiplying the denominator (bottom) of one fraction by the denominator of the other fraction (NCTM, 2010). So, in this case, the numerator 1 of $\frac{1}{2}$ is multiplied by the numerator 1 of $\frac{1}{3}$, giving 1, and the denominator 2 of $\frac{1}{2}$ is multiplied by the denominator 3 of $\frac{1}{3}$, giving 6. The final result is $\frac{1 \times 1}{2 \times 3} = \frac{1}{6}$ (Fosnot & Dolk, 2002).

The results of this research show that all students are able to carry out the multiplication operation procedure on fractions correctly. However, none of them really understood the meaning of the symbol '×' in the context of fractional numbers. This is because they are fixated on the meaning of multiplication which is always interpreted as repeated addition as in the following interview excerpt with one of the respondents.

R : "If you interpret 3×4 in the previous problem as the repeated addition of 4+4+4, how do you explain the meaning of the symbol '×' in this problem?"
S1 : “multiplication of fractions”
R : “How do you interpret it?”
S1 : "Combining the fraction $\frac{1}{2}$ with the fraction $\frac{1}{3}$ becomes the result of multiplying the fraction $\frac{1}{6}$"
R : “How Do You Put Them Together?”
S1 : "multiplied by each denominator and numerator, namely 1 times 1 the result is 1 and 2 times 3 the result is 6"
R : “Why do you multiply them? Didn't you just say 'combine them'?”
S1 : "Yes, that has been combined into the fraction $\frac{1}{6}$"
R : "Doesn't combining mean adding up so it should be $\frac{1}{5}$?"
S1 : "mmmm.."

Failure to understand the symbol '×' in multiplying fractions can be categorized as a 'conceptual gap' which according to Hiebert and Lefevre (1986) can occur when there is a gap between the operational procedures that a person has mastered and the underlying conceptual understanding. For example, although students are able to calculate the result of $\frac{1}{2} \times \frac{1}{3}$, they do not understand that this operation is not simply repeated addition, but rather the division of smaller units from a whole (NCTM, 2010; Wu, 2011).

This misunderstanding can be caused by the limited learning experience of students who focus more on memorizing rules without a deep understanding of the broader mathematical meaning of multiplication operations, especially in the context of multiplying fractions which requires a deeper understanding of parts of the whole (Fosnot & Dolk, 2002). In this case, students may fail to relate the meaning of the symbol ‘×’ to more complex concepts such as division and the scale of units in fractions, which are clearly different from the repetition of simple addition (Tall & Vinner, 1981).

Problem 3

Heid (2005) states that symbolic representation is an essential tool for understanding and solving complex mathematical problems. According to Kaput (1992), symbolic representation is not only a communication tool in mathematics, but also a thinking tool that is very important in developing mathematical concepts in depth. Appropriate symbol representation helps students build a deep and flexible understanding of arithmetic operations that can later be applied in various contexts (Hiebert & Lefevre, 1986). In addition, a comprehensive understanding of symbol representation also allows prospective teachers to teach mathematical concepts more effectively and helps students develop critical thinking and problem-solving skills (Sfard, 1991). Therefore, exploring students' understanding of the representation of the symbol '×' is a crucial step in equipping them with the knowledge and skills necessary to become competent educators and able to teach mathematics in a meaningful and relevant way (Van de Walle et al., 2019). Problem 3 is used to explore the ability to represent the symbol '×' because this problem not only explores how student teachers understand and explain various representations of the symbol '×' in the context of multiplication, but also how they can communicate these concepts to elementary school students in a way clear and easy to understand.

We found that although the task seemed simple, not many students were able to correctly create four different representations of 3×6. This reflects the conceptual difficulties or representational understanding difficulties that are often encountered in basic mathematics learning. The term representational understanding difficulty refers to a person's difficulty in understanding or representing certain information or concepts in a more abstract or symbolic form, such as graphs, diagrams, or mathematical representations (Carpenter et al., 1996; Lesh et al., 2003). These difficulties can arise due to limitations in their cognitive abilities to process information abstractly or symbolically, difficulties in understanding symbols or mathematical representations used in certain contexts, lack of experience or practice in using or interpreting certain representations, and difficulties in transferring knowledge from one form to another. representation to other forms of representation.

Representing 3×6 using an Array (organizing objects in rows and columns) can be done by describing several objects into 6 columns and 3 rows as in Figure 5.

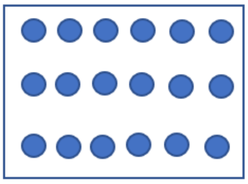


Figure 5. Array representation of 3×6

This representation has 6 groups or columns of elements placed vertically with each column containing 3 elements. Only 27 student respondents wrote it as 6 columns and 3 rows, 22 respondents described the opposite, namely 3 columns and 6 rows in the array (Figure 6), while the other 2 did not answer.

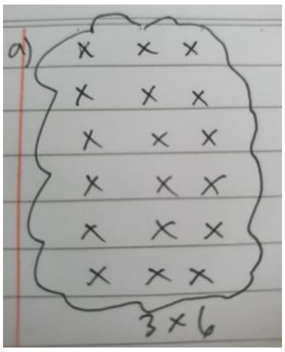


Figure 6. Example of 3×6 array student representation

In the array representation of multiplication, the answer can be found by counting the number of items. Of course, there is no difference in the results, as Figure 5 and Figure 6 will both produce 18. However, it is best to use one method (the first number as a row and the second number as a column) for each question at the beginning to avoid confusion as well as provide consistent representation. This also makes it easier to relate to the other three ways of representation. Apart from that, the number of rows and columns in an array affects how to read and interpret the data or information that is represented, and also affects how the data is processed in analysis or mathematical operations. For example, in matrix operations, the number of rows and columns determines the dimensions of the matrix and the rules that apply in operations such as addition, multiplication, and so on.

The representation of multiplication with equal groups (groups of the same size) is very closely related to the concept of repeated addition because it describes how a certain number can be produced by grouping elements into groups that have the same number. Therefore, the representation of multiplication with equal groups provides a concrete understanding of how multiplication is a form of repeated addition which is very important to help strengthen the conceptual understanding of basic arithmetic operations. This makes it easier for students or individuals to understand that multiplication is a more efficient way to add repeatedly the same number, so they can apply this concept in more complex mathematical contexts.

Representing 3×6 by means of an equal group, it can be visualized as three groups each consisting of six elements. Each of these groups essentially shows the repetition of adding the number 6 three times $(6 + 6 + 6)$ as in Figure 7.

Figure 7. Represents 3×6 in an equal group

A total of 10 respondents represented using Figure 7 and 41 others represented the opposite (6 groups each consisting of three elements).

In the repeated addition section, which is supposed to be the most basic, many students are still confused and tend to just repeat the multiplication operation without understanding the essence of the repetition as addition of the same group. Even more surprising, the number line representation, which depicts multiplication as a jump on a number line, seems to be the biggest challenge or something they have never encountered before.

Multiplication on a number line is the application of the multiplication operation to a certain set of numbers on a number line. Multiplication is also called repeated addition. To perform multiplication on a number line, start from zero and move towards the right side of the number line several times. In 3×6, start from zero then form three groups of 6 equal intervals on the number line reaching 18. Look at Figure 8 below which shows $3 \times 6 = 18$.

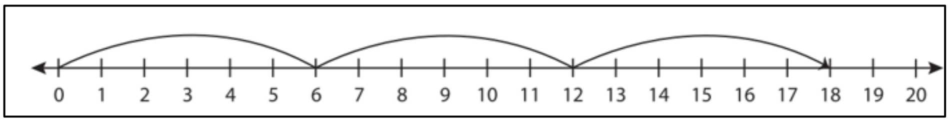


Figure 8. Multiplication $3 \times 6 = 18$ on a number line

A total of 45 students experienced difficulty in representing the multiplication $3 \times 6 = 18$ on a number line for several reasons. First, the concept of a number line is often easier to understand in the context of direct addition or subtraction, where each step or jump along a number line involves only one number unit. However, when it comes to representing multiplication, such as 3×6, students need to understand that they have to make several big leaps, not just one small step. This requires a deeper understanding of how the jump is repeated continuously and leads to the correct endpoint. For example, to represent 3×6, they had to make three jumps of 6 units on the number line, which ultimately took them from 0 to 18. However, only one of the 51 respondents did it correctly. Another seven respondents had already made three jumps, but unfortunately, they didn't start from zero, which is an important step in understanding this concept properly. More than 80% of respondents, namely 42 students, described six jumps without clear jump direction markings and also did not start from zero, indicating a fundamental misunderstanding of how repeated jumps on a number line should be performed to represent multiplication. Meanwhile, another respondent only drew one big jump from the numbers 1 to 18, as seen in Figure 9, which is also a wrong approach and does not show a correct understanding of the concept of multiplication on a number line.

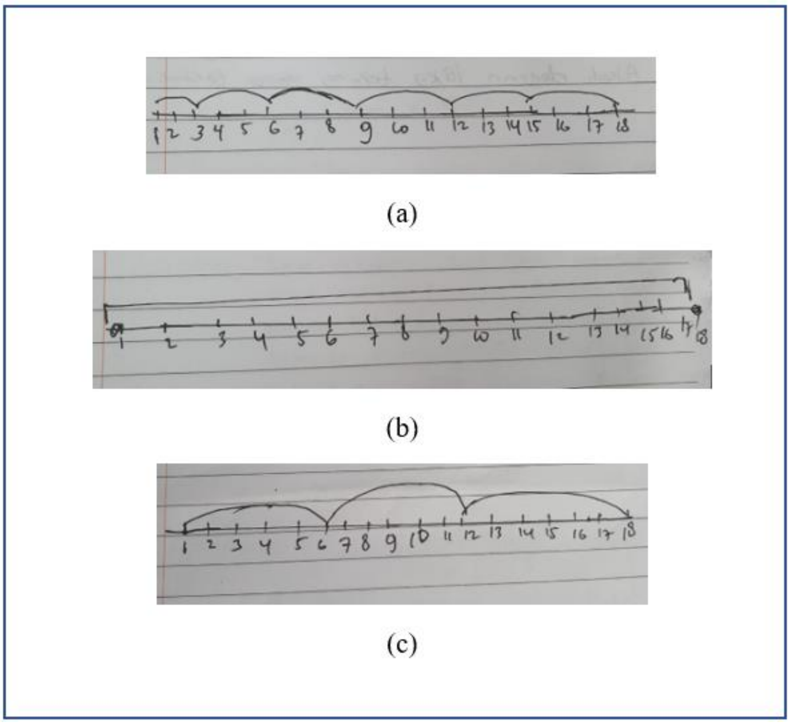


Figure 9. Example of an error in representing 3×6 on a number line

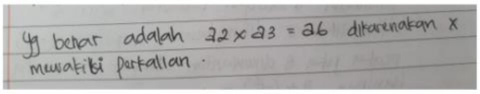
These difficulties are often exacerbated by a lack of visualization or direct experience in using the number line for more complex operations such as multiplication, as well as thinking habits that are still dominant in the context of ordinary addition. This underscores the need for more effective teaching and clearer visualization in helping students understand and represent basic mathematical operations such as multiplication on a number line.

In the representation of multiplication on a number line, the number of jumps, the direction of the jumps, and the starting point are very important to understand correctly. The number of jumps represents the first factor in the multiplication; for example, in 3×6, the number 3 represents three jumps of 6 units which describe the repeated addition of the number 6. If the number of jumps does not match, the result obtained will be wrong and will not describe the correct product. The direction of the jump indicates the nature of the number involved, where a jump to the right indicates positive repeated addition, and to the left indicates subtraction or addition with a negative number. This is important when doing multiplication with negative numbers such as $3 \times (-6)$ or $(-3) \times (-6)$. Starting from zero is important as a basic reference to ensure each unit is calculated correctly. This helps in understanding the progression of numbers from a neutral point. Starting from another point may result in errors in determining the final result due to an incorrect start of the calculation. A clear understanding of these elements ensures that multiplication can be represented and understood correctly, reinforces concepts, and helps prevent conceptual errors that can hinder understanding of more complex mathematics and its application to real problems (Van de Walle et al., 2019).

Problem 4

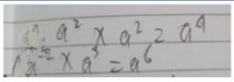
The ability to make connections between concepts is a critical skill in deep understanding of mathematics, because it allows students to see the relationships between various elements and principles, making it easier for them to apply knowledge in different contexts (Tall, 2008). Problem 4 is used to explore students' understanding of connections between concepts, especially in the use of the symbol '$\times$' and the laws of exponents. In this problem, student teachers must understand that the symbol '$\times$' is not only a common multiplication symbol, but also indicates that when we multiply two exponents with the same base, the exponents must be added (so $a^2 \times a^3 = a^5$). This forces students to connect the concept of multiplication with the concept of addition of exponents, as well as deepening their understanding of how mathematical operations function at different levels of abstraction. Through these questions, students are challenged to differentiate between operating rules that apply to ordinary numbers and those that apply to exponents, which in turn strengthens their skills in transferring and applying concepts in a variety of mathematical situations.

Mastering the concept of exponents is very important for understanding and mastering exponents (Smith III et al., 1994; Tall, 2008; Van de Walle et al., 2019). A good understanding of exponents prevents conceptual errors such as misunderstanding that exponents must be multiplied when multiplying numbers with the same base such as $a^m \times a^n = a^{m+n}$, not $a^m \times a^n = a^{m \times n}$. Mastering exponents helps in avoiding these mistakes and ensures proper calculations. Of the 51 prospective student teachers who were respondents, there were 4 students who answered $a^2 \times a^3 = a^6$, an error known as an error in the exponent rule or misapplication of the exponent rule (Van de Walle et al., 2019). Figure 10 shows the respondent's answer $a^2 \times a^3 = a^6$along with the reasons.



The correct one is $a2 \times a3 = a6$ because × represents multiplication.

(a)

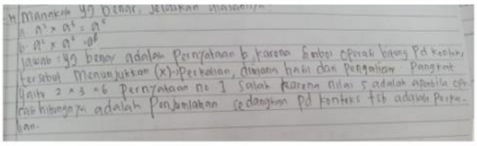


$a^2 \times a^2 = a^4$ so $a^2 \times a^3 = a^6$

(b)

$a2 \times a3 = a6$ because when an equation has variables a2 and a3, you can combine them by multiplying both sides of the first equation by $a2 \times a3 \times a3 = a5 \times a3a2 \times a2 \times a3 = a6 \times a2$.

(c)



The statement ($a^2 \times a^3 = a^6$) is correct because the arithmetic operation symbol in this context shows (×) multiplication where the result of multiplying the powers is $2 \times 3 = 6$. Statement no. 1 is wrong because the value 5 is if the operation is addition, whereas in that context it is multiplication.

(d)

Figure 10. Example of answer $a^2 \times a^3 = a^6$ and the reasons

This error occurs because they misunderstand those exponents must be added when multiplying two numbers with the same base, not multiplied as in the correct answer, namely $a^2 \times a^3 = a^5$ (Tall, 2008). This misapplication reflects a fundamental lack of understanding of exponential operations and the importance of following precise mathematical rules to avoid incorrect results (Smith III et al., 1994).

Problem 5

Exploring problem solving skills in prospective elementary school teacher students is very important in exploring the concept image of the '×' symbol because it helps them understand the diverse meanings and applications of this operation in various contexts. By developing problem solving skills, students not only learn that the symbol '×' means simple multiplication, but also see the relationship between this symbol and other mathematical concepts thereby expanding their understanding of how mathematical operations can be used to solve various types of problems (Tall, 2008). This skill also allows them to differentiate the use of the symbol '×' in different contexts and to avoid conceptual misunderstandings that could hinder their students' future learning (Van de Walle et al., 2019).

Problem 5 trains the ability to apply appropriate mathematical rules to calculate the final result of this sequence of operations, as well as honing critical skills. Thus, students learn how the symbol '×' is used in the context of solving simple mathematical problems, expanding their understanding of basic mathematical operations, and their application in everyday life (Van de Walle et al., 2019). This kind of questions also help students to develop their ability to provide clear and systematic explanations to their future students, thereby increasing the effectiveness of mathematics teaching at the elementary level (Skemp, 1987).

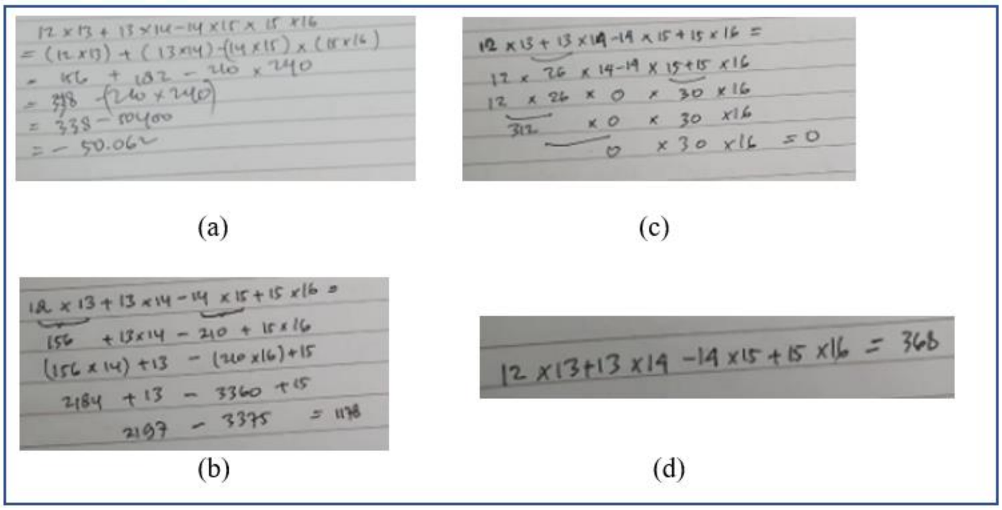


Figure 11. Various errors in problem 5

Figure 11 (a) shows that the respondent was not careful in reading the question, namely writing 15 + 15 as 15 × 15, resulting in inaccurate calculation results. Figure 11 (b) shows the errors in the third step, namely $(156 \times 14) + 13$ and $(210 \times 16) + 15$. In this context, it seems that the student respondents have understood the order of operations, namely grouping similar operations and prioritizing multiplication operations over addition operations. However, the steps taken were inappropriate and did not comply with procedures. There were at least 3 students who made similar mistakes. Figure 11 (c) errors occur from the first step when prioritizing addition over multiplication. There was 1 respondent who made this mistake. Meanwhile, in Figure 11 (d), the respondent performs sequential operations from left to right, such as reading text and ignoring the order of calculation operations in arithmetic. These errors are a form of sequence of operations errors or errors in following arithmetic operation procedures that arise when respondents do not follow the correct sequence of mathematical operations, which often results in inaccurate calculation results.

Correct interpretation of symbols is very important because misinterpretation of symbols can cause conceptual errors that have a significant impact on the results of arithmetic operations, often resulting in incorrect solutions and confusion in mathematical understanding (Önal, 2023; Schoenfeld, 2011; Skemp, 1987). When symbols are not understood correctly, such as the symbol '×' which is often only understood as repeated addition without understanding the wider context, this can lead to the application of incorrect mathematical rules and calculation errors (Tall, 2008). Errors in interpreting symbols often reflect deficiencies in understanding basic concepts, which can reinforce misconceptions and hinder learning progress (NCTM, 2010).

This research has provided in-depth insight into the perceptions and understanding of prospective elementary school teacher students towards the symbol '×' in the context of arithmetic which was explored through a hermeneutic phenomenology approach. Our findings show that students' concept image of the symbol '×' is strongly influenced by learning and teaching experiences which are often limited to a procedural view and pay less attention to deeper and more applicable concepts (NCTM, 2010; Tall, 2008). Students tend to see the symbol '×' solely as a tool for basic operations without understanding broader conceptual relationships, such as distribution or multiplication of fractions, which can cause conceptual errors known as conceptual gaps (Hiebert & Carpenter, 1992). This research emphasizes the importance of conceptual understanding in mathematics teaching, where a more exploratory and contextual approach is needed to strengthen the concept image of prospective teacher students (Boaler, 2016; Skemp, 1987). This research also highlights how personal experiences and educational background play an important role in shaping students' understanding of mathematical symbols (Mason & Spence, 1999; Schoenfeld, 2011). This shows the need for a more inclusive and varied curriculum that not only focuses on procedural skills, but also encourages a broader and more contextual understanding of mathematics.

Practical applications of these findings can be used to design more effective teacher training programs that emphasize in-depth conceptual understanding and teaching skills of basic mathematical concepts. In addition, the results of this research can help in developing more comprehensive teaching materials and textbooks so that they do not only focus on basic operations but also on the application and exploration of broader concepts (Van de Walle et al., 2019). Further research needs to be conducted to explore the relationship between conceptual understanding and application of mathematics in more diverse contexts. This research could include longitudinal studies that track the development of students' mathematical understanding from the beginning to the end of their studies, as well as research on the effectiveness of various teaching approaches in improving conceptual understanding (Hiebert & Grouws, 2007). Thus, this research not only provides insight into current perceptions, but also offers directions for broader improvements in mathematics education in the future.

Learning Experience

The students' learning experience shows insignificant diversity, because the symbol '×' is usually introduced only as repeated addition, such as $2 \times 3 = 3 + 3$, and then continued with the multiplication table to be memorized. Therefore, there is no specific sample related to a deeper learning experience. As a result, students often only have an understanding of the symbol '×' in the context of integer operations and have difficulty in explaining the meaning of the concept of multiplication in other contexts, such as in the multiplication of fractions, for example $\frac{1}{5} \times \frac{1}{3}$.

In this study, several factors were found that caused students' limited understanding of the concept of the '×' symbol. First, students tend to interpret the '×' symbol only as a sign of repetition or the total number of the same group, so they see it only as a tool for calculating the results of multiplying integers without understanding its application in other contexts such as fractions or negative numbers. In addition, students are not aware of the conceptual differences between 3 × 4 and 4 × 3, because they only focus on the same result without understanding that the order of numbers can have different meanings in certain contexts. Difficulty in understanding multiplication with negative numbers is also seen, where students only rely on memory that the results of negative and positive multiplication are negative, without understanding the basic concept behind this rule. Although they are able to perform multiplication operations on fractions correctly, they do not understand the meaning of the '×' symbol in the context of fractions because they are trapped in the interpretation of multiplication as repeated addition. In addition, errors in visual representation, such as the inability to represent arrays or number lines consistently, indicate a shallow or inconsistent understanding of the meaning of multiplication. Finally, errors in exponent rules, such as answering $a^2 \times a^3 = a^6$, indicate a lack of in-depth understanding of the concept of multiplication operations in algebra. These findings suggest that students' understanding of the symbol '×' tends to be limited and contextual, with a strong focus on repeating integers and difficulty in applying multiplication concepts to more complex situations.

Learning experiences such as excessive use of calculators can make them only get used to the final results without understanding the processes behind mathematical operations (Leung & Bolite-Frant, 2015). Learning that only focuses on mechanical procedures and the use of symbols in a narrow context also strengthens the operational view, where the symbol '×' is seen only as a simple multiplication operation, without exploring its application in various contexts such as multiplying fractions or distributions (Van de Walle et al., 2019). Failure to understand the symbol '×' in the context of multiplying fractions indicates a significant conceptual gap, where their understanding of basic mathematical concepts is not applied correctly and consistently in more complex situations (Schoenfeld, 2011). This difficulty is often categorized as representational understanding difficulty, where they experience difficulty in visualizing and applying the concepts learned in various mathematical representations (Mason & Spence, 1999). Errors in the exponent rule or misapplication of the exponent rule also indicate a lack of deep understanding of the properties of exponents and how they are applied in different contexts (Tall, 2008). As a result, when faced with problems that require understanding the correct sequence of operations, they tend to make errors in the sequence of operations or procedural errors which often result in inaccurate calculation results (Skemp, 1987). Learning experiences that do not provide space for concept exploration and deeper understanding, as well as learning that is too focused on using tools such as calculators without a strong basic understanding, contribute to the formation of a limited concept image and cause conceptual errors in mathematics.

CONCLUSIONS

The conclusion of this study reveals that the understanding of elementary school student teachers towards the symbol '×' in the context of arithmetic is still limited and often procedural, with little attention to deeper and more applicable concepts. Students tend to see this symbol only as a tool for basic operations, without understanding broader conceptual relationships, such as in the context of distribution or multiplication of fractions. This indicates a conceptual gap that can hinder the development of a more holistic understanding of mathematics. To overcome this problem, a more exploratory and contextual teaching approach is needed, as well as a curriculum that emphasizes not only procedural skills but also broader conceptual understanding. By strengthening students' concept images and developing more effective teacher training programs, we can ensure that student teachers have a deep understanding and are able to transfer this understanding to their students in the future. This study provides an important basis for improvements in mathematics education and paves the way for further research that can address the gap in conceptual understanding among student teachers.

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